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AUTHORS' REPLY

The Letter to the Editor by Rempfer makes two claims regarding inconsistencies and errors in the paper by Sani *et al.* [1]. Our response to these claims are as follows.

1. *OUR PAPER IS INCONSISTENT*
   His argument is that it contradicts a conclusion made in [2]. This criticism is irrelevant to [1]. Furthermore, the claims in [2] were made on a rather informal basis in which case, just like a conjecture, one can later publish results that contradict a conjecture without having to be accused of being wrong.
2. *THEOREM* 1 *IN* [1] *IS WRONG*
   Rempfer said that we forgot to add a hypothesis that $\nabla\cdot u$ is continuous in time. This is not so. The same proof as in our paper [1], with more details, is presented at the end of these remarks.
   He also claims that the results in [1] must be wrong because he had 'proved' in Corollary 1 of his earlier paper [3] that the S-CPPE is ill-posed. However, his proof of this corollary is not correct; it cannot be derived from Theorem 2 in [3] since a solution u of the modified problem with Equation (24) in [3] is not necessarily a solution of Equation (30) in [3] due to the fact that $\nabla\cdot u$ is not necessarily zero.

Further, we need to point out some other inconsistencies in [3]:

1. In Lemma 1 in the Appendix, the regularity result quoted from Heywood and Rannacher [4] does not appear in that paper in the form stated in [3]. Actually, it is known to be incorrect without additional compatibility conditions on the initial data.
2. The main point of that 1982 paper [4] in the discussion concerning the regularity of solutions of the Navier–Stokes equations,

$$\partial_t u - \nu\Delta u + u\cdot\nabla u + \nabla p = f \tag{1}$$

$$\nabla\cdot u = 0 \tag{2}$$

   at $t=0$ is completely missed. This is due to the imprecise notion of regularity of solutions of the Navier–Stokes problem used in [3]. In the Heywood and Rannacher paper [4] it is shown that the corresponding weak solution has a limit $p_0(x)=\lim_{t\to 0} p(x,t)$, which is the weak solution of the boundary value problem (for homogeneous boundary data $u|_{\partial\Omega}=0$)

$$\Delta p_0 = -\nabla\cdot(f(\cdot,0) - u_0\cdot\nabla u_0) \quad \text{in } \Omega \tag{3}$$

$$\partial_n p_0 = n\cdot(\nu\Delta u_0 + f - u_0\cdot\nabla u_0) \quad \text{on } \partial\Omega \tag{4}$$

   However, the convergence of the corresponding tangential component

$$\partial_\tau p_0 = \tau\cdot(\nu\Delta u_0 + f - u_0\cdot\nabla u_0) \quad \text{on } \partial\Omega \tag{5}$$

   is only obtained under higher regularity requirements on the weak solution $\{u, p\}$ for $t\to 0$, which results in the well-known non-local compatibility conditions for the initial data $u(0)$. This different behavior of normal and tangential components is the essential point of the regularity analysis in [4] and is missed in the arguments in [3]. The result in [3], which is claimed to contradict, and therefore falsify, the result in [4] is established under the strong assumption of a smooth behavior of the solution $\{u, p\}$ of the Navier–Stokes problem as $t\to 0$. Hence there is actually no conflict at all between the

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two statements and the confusion is due to an improper notion of regularity for weak solutions of the Navier–Stokes problem used in [3].

To see clearly the difference in 'regularity' discussed above, see the 'Impulsive Start' discussion in [5, Section 3.19, p. 884]. There an example is presented both ways: (1) the tangential component of velocity at the wall suffers a discontinuity at $t=0$ and (2) the tangential component (as well as the normal component, of course) is continuous as $t\to 0$. Both solutions are mathematically legitimate AND it is shown that the latter tends to the former, with both generating vortex sheets, as its time constant tends to zero. Vortex sheets comprise an important concept that is not allowed according to Rempfer's theory. Also true for both is that $\nabla\cdot u$ is continuous at $t=0$.

PROOF

Notation

Classic formulation of the transient Stokes problem: with $V=\{v\in H_{1_0}(\Omega)^d:\nabla\cdot v=0\}$ and $f\in L_2(0,T;H^{-1}(\Omega)^d)$ there is a unique $u\in L_2(0,T;H_{1_0}(\Omega)^d)$ and a unique $p\in L_2(\Omega\times(0,T))/R$ such that

$$\int_{\Omega\times(0,T)}[\partial_t u\cdot v+\nu\nabla u:\nabla v+\nabla p\cdot v-f\cdot v]=0 \quad \forall v\in L_2(0,T;H_{1_0}(\Omega)^d) \tag{6}$$

$$\int_{\Omega\times(0,T)} q\nabla\cdot u=0 \quad \forall q\in L_2(\Omega) \tag{7}$$

The new formulation finds $u\in L_2(0,T;H_{1_0}(\Omega)^d)$, $p\in L_2(\Omega\times(0,T))/R$ with

$$\int_{\Omega\times(0,T)}[\partial_t u\cdot v+\nu\nabla u:\nabla v-p\nabla\cdot v-f\cdot v]=0 \quad \forall v\in L_2(0,T;H_{1_0}(\Omega)^d) \tag{8}$$

$$\int_{\Omega\times(0,T)}[(p-\nu\nabla\cdot u)\Delta q-q\nabla\cdot f]=0 \quad \forall q\in L_2(0,T;H_2(\Omega)^d) \tag{9}$$

Theorem

The solution of (8)–(9), if any, is solution of (6)–(7).

Proof

Take $v(x,t)=\nabla\phi$, $\phi\in L_2(0,T;H_{2_0}(\Omega)^d)$ in (8). Then

$$\int_{\Omega\times(0,T)}[\partial_t u\cdot\nabla\phi+\nu\nabla u:\nabla\nabla\phi+\nabla p\cdot\nabla\phi-f\cdot\nabla\phi]=0 \quad \forall\phi\in L_2(0,T;H_{2_0}(\Omega)^d) \tag{10}$$

Now integrating by parts, it is found that

$$\int_{\Omega\times(0,T)}[\partial_t u\cdot\nabla\phi-(p-\nu\nabla\cdot u)\Delta\phi+\phi\nabla\cdot f]=0 \quad \forall\phi\in L_2(0,T;H_{2_0}(\Omega)^d) \tag{11}$$

Now by (9) and an integration by parts in the first term of (11) we find

$$\int_{\Omega\times(0,T)}(\partial_t\nabla\cdot u)\phi=0 \quad \forall\phi\in L_2(0,T;H_{2_0}(\Omega)^d) \tag{12}$$

By taking $\phi(x,t)=a(t)b(x)$ we obtain

$$\int_0^T a(t)\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(\nabla\cdot u)b(x)=0 \quad \forall b\in H_{2_0}(\Omega)^d \;\; \forall a\in L_2(0,T) \tag{13}$$

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Let $c(t)=\int_\Omega(\nabla\cdot u)b$; by definition $c\in L_2(0,T)$, then (13) says that $\mathrm{d}c/\mathrm{d}t=0$ in $L_2(0,T)$ so $c\in H_1(0,T)$. Functions in H_1 are continuous in one dimension of space and $c(0)=0$ so $c(t)=0$ for all t. □

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